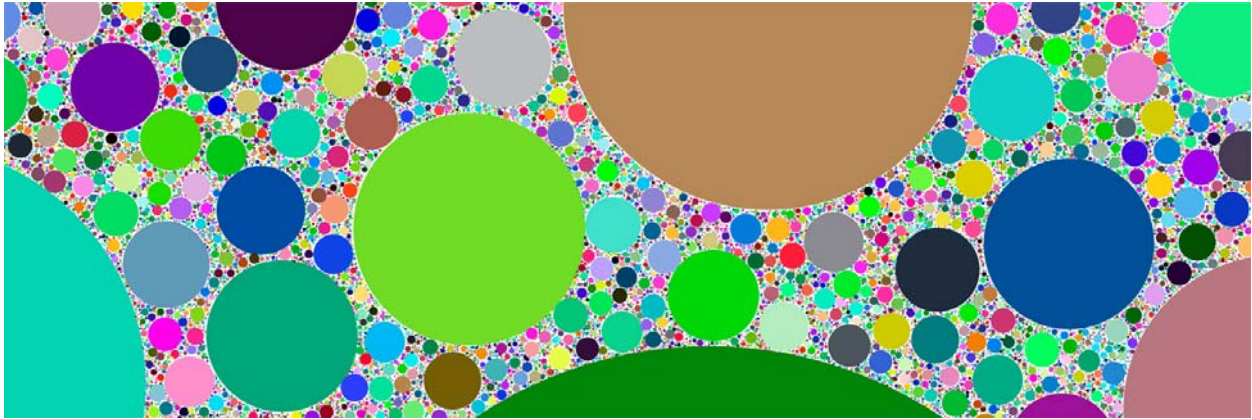


# Statistical Geometry Compared.

## 1. Introduction.



The picture illustrates "statistical geometry" for the particular case of circles. It is taken from a much larger image containing around 660000 circles. It is space-filling<sup>1</sup> in the limit where the process is allowed to run "to infinity". It is assumed that the reader is familiar with this. The rules for construction are stated formally (for 2D) in the Appendix. The algorithm that makes such patterns is here called *statistical geometry* in agreement with the author's previous usage. There is much more information and many examples of this on the "links" page at [john-art.com](http://john-art.com).

The resulting patterns are fractals. The best information currently available says that almost any shape can be "fractalized" in this way -- it isn't just circles. The algorithm works equally well in 1, 2, and 3 Euclidian dimensions. It has parameters  $c$  and  $N$  which affect its properties.

The purpose of this essay is to compare this object with the various known forms of space-filling mathematically-generated patterns.

I do not call the results of the algorithm a "set" because it is more than that. I use the word "object" for lack of a better one. The image only shows one of the infinitely many possible arrangements of *the same circles*. The creation of the sequence of circle sizes is a deterministic process, while their placement is random. Thus the object of interest is really a huge, sprawling thing -- an infinite sequence of circles arranged in an infinite number of ways, all of these ways belonging to the same "object".

## 2. Tessellations.

Ways to cover a finite region of space with same-shape "tiles" have been known since antiquity. In the 19th century it was proved mathematically that there are just 17 possible forms of periodically repeating symmetry for such tilings. In the 20th century a number of nonperiodic tilings have been found by Roger Penrose and others. Perhaps the biggest distinction between these space-filling constructions and statistical geometry is that they are able to fill a spatial region with a *finite* number of tiles (or shapes). Tessellations are not fractal. Statistical geometry does not have any of the rotation, translation, etc. symmetries which tessellations have.

## 3. Circle Packing.

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<sup>1</sup> There is no formal proof of this (i.e., that the algorithm continues indefinitely without stopping), but for discussion purposes this and several other properties of the algorithm are assumed to be true.

This classical technique goes back to the Greek geometer Apollonius and has a large mathematical literature. Apollonius showed that given three circles one can always find another circle tangent to all three of them. This is used in the construction of circle packings by starting with a fixed shape, "seeding" it with a few artfully placed circles, and then placing ever-smaller non-overlapping mutually-tangent circles within the interstices between triplets of previously-placed circles. It is thus space-filling in the limit as one places an *infinite* number of circles -- a property it shares with statistical geometry. Circle packings of this kind are fractal [1].

Perhaps the biggest difference between this and statistical geometry is that circle packing is deterministic. Given a particular seeding arrangement, anybody who does the construction gets the same result, i.e., it is non-random. The "gasket" of space left over after placing some number of circles is in principle a disconnected collection of concave regions, while for statistical geometry it is a single<sup>2</sup> connected region. Circle packing has no parameters such as  $c$  and  $N$  for statistical geometry. It uses only a single shape -- the circle -- and relies upon its properties. Circle packings have symmetries such as rotation, inversion, and mirror planes, while statistical geometry lacks them.

#### **4. Random Apollonian Packing.**

This subject seems to have been pursued largely by physicists [3-5]. It is part of the trend toward fractal models seen in physics beginning in the late 20th century. The constructions are viewed as models for fragmented matter such as sand and gravel, and it thus makes sense to insist that the shapes touch each other since they would do so if made of real matter.

In these constructions one begins by seeding the chosen area with suitably placed "starter" shapes. The algorithm then identifies interstitial locations for the placement of new shapes and fills them with tangent (usually smaller) shapes by a rather complicated algorithm. This results in a list of shape sizes and positions, with both the size and the position having randomness.

In statistical geometry the positions have randomness, but the sizes are dictated by a deterministic sequence.

The authors of these papers assume that there is a unique relationship between the shape of the elements placed and the power-law exponent for their distribution (and thus a relationship with the fractal dimension). They proceed to find values for these exponents under various assumptions. There are thus no user-specified parameters such as  $c$  and  $N$  for statistical geometry.

Based upon the illustrations for [4-5] this method would appear to be usable with a wide variety of shapes (like statistical geometry). It could probably accommodate mixed shapes, although the authors haven't tried that.

#### **5. Statistical Physics.**

This is a very large body of theory [6] going back to Maxwell, Boltzmann, and Gibbs. The most comparable example would be a "hard sphere gas". In such a gas the shapes are all the same, but their configuration (i.e., spatial arrangement) varies randomly. Thus it shares with statistical geometry the property of having a huge number of random configurations of the shapes. Statistical physics defines an "ensemble" of a huge number of such configurations and calculates "ensemble averages" as averages over configurations where the number of them goes to infinity.

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<sup>2</sup> In computations with ordinary floating-point numbers the shapes will sometimes touch (or overlap) due to roundoff error. With infinite-precision "pure math" numbers it is less clear. Touching (for the circle case) would require the distance between two infinite-precision random coordinate pairs ( $x$  and  $y$  values) to agree exactly with the infinitely precise sum of the two radii. The probability of this would appear to be zero. A related question is whether the circles are to be viewed as closed or open sets, something the author is ill-qualified to judge.

Such ensemble averages would appear to have a part to play in statistical geometry. One could ask, for example: "What is the ensemble-average number of trials needed to place the  $n$ -th shape?" Finding values or formulas for such averages in statistical geometry would appear to be a formidable undertaking.

Statistical physics defines a number of statistical properties such as density, energy, temperature, entropy, etc. Such definitions are lacking in statistical geometry, although one wonders if some may eventually be found. Some statistical physics properties such as momentum appear to have no analog in statistical geometry.

Statistical physics treats huge numbers of identical elements (atoms, molecules) with (often) only the coordinates being random. In statistical geometry every element (circle, square, etc.) is different -- in size, shape, etc.

Statistical physics configurations are not fractal<sup>3</sup> in the usual sense.

Many of the results in the statistical physics of gases are independent of the shape, size, and properties of the unit element (a molecule).

The degree of order in a statistical geometry fractal depends greatly on the parameter  $c$  (the area power law exponent). With increasing  $c$  values the distribution is more regular and the average spacing between shapes is smaller relative to the size of the last-placed shape. It can be speculated that  $c$  plays a role like the "order parameters" of statistical physics.

In the statistical physics of gases the density is relatively low and the atoms (e.g., hard spheres) are free to move throughout the volume of interest. In liquids the atoms are much more constrained, but still free to move. This is quite unlike Apollonian constructions where the circles are so constrained that they cannot move.

One can wonder if statistical geometry has such a property. Suppose one has two fractal sets of circles created with the same  $c$ ,  $N$ . Is it possible to move all the circles in set A to their positions in set B by continuous motions<sup>4</sup> in the plane without causing any two circles to overlap?

Can the random statistical geometry circle fractals be viewed as a kind of "gas phase" and the Apollonian circle packings as a kind of "crystalline phase" of fractal circles?

## 6. Summary.

The best information presently available shows that statistical geometry is new and unique in many of its properties.

Viewed as mathematics it combines both the precise shapes of geometry and the randomness of statistics. Study of its properties would appear to need mathematicians who are knowledgeable in both of these areas. Are its regularities to be embodied in precise theorems, or in the determination of probabilities? Or both? The version for a single Euclidian dimension may be the best place to start for fundamental mathematical studies.

Viewed as physics, about all one can say at the present time is that it is a model in search of a system. It may be the simplest existing fractal model which fills space randomly with an infinite number

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<sup>3</sup> They do show fractal long-range order near a phase transition point.

<sup>4</sup> By this we would mean continuous nonoverlapping motions of *all* of the circles. It is obvious in most cases that a circle in A can't be moved to its place in B without moving many circles during the process. This idea may seem paradoxical. "At infinity", isn't space 100% full? In other of my writings I have shown that at any stage of the fractalization process the average gasket width is a roughly fixed fraction of the linear dimension of the next-to-be-placed shape. This fraction comes closer and closer to a particular value as  $n \rightarrow \infty$ . The smallest shapes always have "wiggle room" at any stage in the process, and there are a huge number of them. The amount of wiggle room depends strongly on the value of  $c$ , and it is quite possible that "shuffling" between configurations is possible for small  $c$  and impossible for large  $c$ .

of shapes. It could be an interesting vehicle for finding how power-law statistical distributions emerge from fundamental considerations rather than simply being empirical findings. The ability to fill all space with a fractal set of shapes may also be viewed as expressing a property of space itself -- space is something that can be fractally divided.

## 7. References.

- [1] "Fractals: Form, Chance, and Dimension", Benoit Mandelbrot, W. H. Freeman & Co. (1977). The *opus magnus* of fractals.
- [2] Circle packing is described many places, including ref. [1]. A good place to begin is the Wikipedia article. Internet searching will disclose many other sources of information.
- [3] P. S. Dodds and J. S. Weitz; Physical Review E<sup>5</sup>, Vol. 65, 056108 (2002).
- [4] P. S. Dodds and J. S. Weitz; Physical Review E, Vol. 67, 016117 (2003).
- [5] G. W. Delaney et al.; Physical Review Letters, Vol. 101, 120602 (2008).
- [6] "Fundamentals of Statistical and Thermal Physics", F. Reif, McGraw-Hill (1965). This is but one example of a huge number of texts and monographs on this subject.

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## Appendix. The Algorithm Stated.

The fractals of interest here are generated using the following algorithmic rules (for 2 Euclidian dimensions):

1. Create a sequence of areas  $A_i$  equal to  $\frac{1}{N^c}, \frac{1}{(N+1)^c}, \frac{1}{(N+2)^c}, \frac{1}{(N+3)^c}, \dots$  where  $c$  and  $N$  are constant parameters. Choose an area (square, rectangle, circle, ...)  $A$  to be filled.
2. Sum the areas  $A_i$  to infinity, using the Hurwitz zeta<sup>6</sup> function

$$\zeta(c, N) = \sum_{i=0}^{\infty} \frac{1}{(N+i)^c}$$

3. Define a new set of areas  $S_i$  by  $S_i = \frac{A}{\zeta(c, N)(N+i)^c}$ . It will be seen that the sum of all these redefined areas is just  $A$ .
4. Let  $i = 0$ . Place a shape having area  $S_0$  in the area  $A$  at a random position  $x, y$  such that it falls entirely within area  $A$ . This is the "initial placement". Increment  $i$ .
5. Place a shape having area  $S_i$  entirely within  $A$  at a random position  $x, y$  such that it falls entirely within  $A$ . If this shape overlaps with any previously-placed shape repeat step 5. This is a "trial".
6. If this shape does not overlap with any previously-placed shape, store  $x, y$  and the shape dimensions in the "placed shapes" data base, increment  $i$ , and go to step 5. This is a "placement".

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<sup>5</sup> Quite a number of such papers can be downloaded from the authors' web sites.

<sup>6</sup> The definitions of the Riemann and Hurwitz zeta functions can be found in Wikipedia. The Riemann zeta function is historically older than the Hurwitz function, and is the special case where  $N = 1$ . In my images  $N$  is usually taken to be an integer, but according to the definition of  $\zeta(c, N)$  it can be any real number  $\geq 1$ . The Riemann zeta function has been much studied in connection with number theory, but it is not evident that such studies have any relevance here.

7. Stop when  $i$  reaches a set number, percentage filled area reaches a set value, or other.

The linear dimensions of the shapes are nowhere specified. They are calculated from the area for the shape of interest. A very wide variety of shapes have been found to be fractalizable in this way. This is a very *simple* algorithm, easily stated in a few lines of text. While the algorithm is simple, it is able to produce complicated<sup>7</sup> results.

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<sup>7</sup> This is a characteristic of many iterative or recursive procedures such as cellular automata and some fractals. Random walk, with its vast number of variations of both form and detail provides another example.